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TWO-PARAMETER EIGENVALUE PROBLEMS IN DIFFERENTIAL EQUATIONS

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ABSTRACT

The problem considered is the following. The parameters λ, μ are to be so chosen that the differential equation

$$\frac{d^2w}{dz^2} + \{\lambda + \mu f(z) + g(z)\} w = 0 ,$$

should have a non-trivial solution satisfying three boundary conditions of the type $w(a) = w(b) = w(c) = 0$.

Formal properties of solutions (orthogonality, eigenfunction expansions) are established, and some results obtained on the real character of the eigenvalues. Two integral relations are given, leading to a reduction of the original problem to a one-parameter problem involving a non-linear integral equation.

TWO-PARAMETER EIGENVALUE PROBLEMS IN DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper a number of inter-related eigenvalue problems are discussed, the common feature being that each involves an ordinary linear homogeneous differential equation, containing two parameters whose values have to be so chosen that the solution will satisfy three boundary conditions. The typical problem is the following:

In the equation

$$\frac{d^2w}{dz^2} + \{\lambda + \mu f(z) + g(z)\} w = 0 , \tag{1.1}$$

the functions $f(z)$, $g(z)$ are given, and constants $\underline{a}$, $\underline{b}$, $\underline{c}$ are also given. It is required to find values of λ, μ such that the equation may possess a solution $w(z)$, not identically zero, satisfying the boundary conditions

$$w(a) = w(b) = w(c) = 0 . \tag{1.2}$$

The treatment of the problem in this paper will be formal only, in the sense that the question of existence of solutions in general will not be discussed.

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This is for two reasons (i) the difficulties involved in establishing existence-theorems appear to be quite formidable, and it seems likely that in order to obtain proofs of such theorems, use will be made of some of the formal properties which will be given here, in particular the connection with certain one-parameter problems in partial differential equations or integral equations (ii) there are several special, non-trivial, examples of problems of the general type considered (or reducible to that type) in which it is known that solutions exist, and a general theory would be of value even if it were found to cover only these.

There are some variants of the main problem which may be noted here.

(a) Any or all of the conditions (1.2) are replaced by the condition that $w'(z)$ should vanish at the corresponding point. The results proved for the main problem in this paper apply with only minor modifications to this case also.

(b) The functions $f(z)$, $g(z)$ are periodic functions with period P, and two of the boundary conditions are replaced by a single periodic condition $w(z+P) \equiv w(z)$.

(c) The functions $f(z)$, $g(z)$ are doubly-periodic, with independent periods P, P', and the three boundary conditions are replaced by a single condition of double-periodicity, $w(z+P) \equiv w(z+P') \equiv w(z)$.

It seems that both (b) and (c) may, in general, need special consideration, but in many of the known cases (Mathieu's and Hill's equations, Lamé's equation and the ellipsoidal wave equation) it is possible to reduce, by a special artifice, the problem to one of the main type or type (a).

Some classical examples should be mentioned here, mainly because of the suggestive illustrations they provide of the various possibilities which may occur.

(A) The problem of the vibrating uniform elliptic membrane (which stimulated the original interest in differential equations with periodic coefficients) gives rise to four two-parameter eigenvalue problems of the form (1.1) with $f(z) = \cos 2z$, $g(z) = 0$, $a = 0$, $b = \frac{1}{2}\pi$, $c = i\alpha$, where α is real and given by $\cosh \alpha = e^{-1}$, e being the eccentricity of the elliptic boundary. In the four problems, we may require $\underline{w}$ or $\underline{w}'$ to vanish at 0 and $\frac{1}{2}\pi$. An equivalent formulation is to require that $w(z)$ have period 2π and vanish at $z = i\alpha$ (4, §4.31).

(B) In the separation of the wave equation in general paraboloidal coordinates we have a problem of the above nature with $f(z) = \cos 2z$, $g(z) = \omega \cos 4z$, (ω being a constant depending on the wave number), $a = 0$, $b = \frac{1}{2}\pi$, and the third boundary condition being either (6)

$$\lim_{u \to \infty} w(iu) = 0 \quad \text{or} \quad \lim_{u \to \infty} w(\tfrac{1}{2}\pi + iu) = 0 .$$

(C) Separation of Laplace's equation in ellipsoidal coordinates gives the Lamé problem in which $f(z) = \mathrm{sn}^2 z$, $g(z) = 0$, $a = 0$, $b = K$, $c = K + iK'$, in the usual elliptic-function notation.

(D) Similar separation of the wave equation gives the same problem with $g(z) = \omega\, \mathrm{sn}^4 z$, ω again being a constant depending on the wave number (3).

2. Reduction to a one-parameter problem in partial differential equations

One method of reducing the original problem to a two-parameter problem is the following.

Let $W(\alpha, \beta) \equiv w(\alpha) w(\beta)$. (2.1)

Then from (1.1),

$$\frac{\partial^2 W}{\partial \alpha^2} + \{ \lambda + \mu f(\alpha) + g(\alpha) \} W = 0,$$

$$\frac{\partial^2 W}{\partial \beta^2} + \{ \lambda + \mu f(\beta) + g(\beta) \} W = 0,$$

hence by subtraction,

$$\frac{\partial^2 W}{\partial \alpha^2} - \frac{\partial^2 W}{\partial \beta^2} + [\mu \{ f(\alpha) - f(\beta) \} + g(\alpha) - g(\beta)] W = 0 , \tag{2.2a}$$

and the boundary conditions become

$$W(\alpha, \beta) = 0 \tag{2.2b}$$

whenever α or β has any of the three values $\underline{a}$, $\underline{b}$, $\underline{c}$.

In the problem posed by (2.2), the parameter λ has been eliminated and only μ is involved. This reduction to a one-parameter problem has been achieved, however, at the cost of replacing the original ordinary differential equation by a partial differential equation. Another obstacle to be overcome should also be noted. A function $W(\alpha, \beta)$ satisfying (2.2a, b) is not necessarily of the "separable" form $w(\alpha)w(\beta)$, and so does not automatically yield a solution of the original problem.

This approach has proved useful in problems (C) and (D) mentioned above. In the former, problem (2.2) has a denumerably-infinite set of eigenvalues μ_n $(n = 0, 1, 2, \ldots)$, such that for $\mu = \mu_n$ there are $(n + 1)$ linearly independent solutions $W_{n,m}(\alpha, \beta)$, $(m = 0 \text{ to } n)$, and precisely $(n + 1)$ linear combinations of these which are of separable form. In problem (D), however, each of the eigenvalues μ_n yields only a single solution which is already separable (7).

3. Orthogonality properties

Let $\int_{z_1}^{z_2} dz$ denote contour integration along a path in the complex z-plane joining z_1 and z_2 and not passing through any singularity of the differential equation (1.1). There may, of course, be more than one such path, but wherever the symbol $\int_{z_1}^{z_2} dz$ occurs, it is presumed to refer to the same path.

If μ in (1.1) is regarded as fixed, the problem reduces to a one-parameter eigenvalue problem, and we have immediately the usual orthogonality property which we shall here call ordinary orthogonality, namely:

Let $w_1(z)$, $w_2(z)$ be solutions of (1.1) and (1.2) for the same μ but different values λ_1, λ_2 respectively of λ. Then

$$\int_a^b w_1 w_2 dz = \int_a^c w_1 w_2 dz = \int_b^c w_1 w_2 dz = 0 \quad . \tag{3.1}$$

This is proved in the usual manner; we write down the differential equations satisfied by w_1, w_2, multiply by w_2, w_1 respectively and subtract, then integrating by parts gives

$$(\lambda_1 - \lambda_2) \int_{z_1}^{z_2} w_1 w_2 dz = [w_1 w_2' - w_1' w_2]_{z_1}^{z_2} \quad , \tag{3.2}$$

and the right hand side vanishes if $(z_1, z_2) = (a, b)$ or (a, c) or (b, c)

on account of the boundary conditions, so the result follows.

In general, one cannot assert that the integral

$$\int_{z_1}^{z_2} \{w_1\}^2 dz \tag{3.3}$$

is non-zero, since we are not excluding complex values of z. If, however, the following two conditions are fulfilled

(i) the path of integration is parallel to either the real or the imaginary axis,

(ii) on the path, $w(z)$ is either real or a pure imaginary throughout,

then clearly

$$\int_{z_1}^{z_2} \{w_1(z)\}^2 dz \neq 0 . \tag{3.4}$$

These conditions - which are sufficient but clearly not necessary - cover the applications in problems (A) - (D) above.

Solutions of (1.1) associated with different values of μ and/or λ satisfy a wider orthogonal relation which, in order to distinguish it from the ordinary orthogonality of (3.1), we shall call <u>double orthogonality.</u>

Let $v_1(z)$, $v_2(z)$ be solutions for different λ and/or μ, i.e. (λ_1, μ_1) and (λ_2, μ_2) with $\lambda_1 \neq \lambda_2$ or $\mu_1 \neq \mu_2$ or both. Let (α_1, α_2) and (β_1, β_2) denote different members of the value-pairs (a, b), (a, c), (b, c). Then

$$\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} v_1(\alpha)v_1(\beta)v_2(\alpha)v_2(\beta)\,\{f(\alpha)-f(\beta)\}\,d\alpha\,d\beta = 0\,. \tag{3.5}$$

<u>Proof</u>

Let

$$V_i(\alpha,\beta) \equiv v_i(\alpha)v_i(\beta)\,, \quad (i = 1, 2.)\,. \tag{3.6}$$

Then by the same reasoning which gave (2.2a)

$$\frac{\partial^2 V_1}{\partial\alpha^2} - \frac{\partial^2 V_1}{\partial\beta^2} + [\mu_1\{f(\alpha) - f(\beta)\} + g(\alpha) - g(\beta)]\,V_1(\alpha,\beta) = 0\,,$$

with a similar equation for $V_2(\alpha,\beta)$. Multiplying these by V_2, V_1 respectively and subtracting gives

$$V_2\{\frac{\partial^2 V_1}{\partial\alpha^2} - \frac{\partial^2 V_1}{\partial\beta^2}\} - V_1\{\frac{\partial^2 V_2}{\partial\alpha^2} - \frac{\partial^2 V_2}{\partial\beta^2}\} + (\mu_1 - \mu_2)\{f(\alpha) - f(\beta)\}V_1V_2 = 0\,.$$

Now integrate over the ranges (α_1, α_2), (β_1, β_2); integration by parts with respect to α or β gives

$$(\mu_1 - \mu_2)\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\{f(\alpha) - f(\beta)\}V_1V_2\,d\alpha\,d\beta =$$

$$= \int_{\beta_1}^{\beta_2}\left[V_2\frac{\partial V_1}{\partial\alpha} - V_1\frac{\partial V_2}{\partial\alpha}\right]_{\alpha_1}^{\alpha_2} d\beta + \int_{\alpha_1}^{\alpha_2}\left[V_1\frac{\partial V_2}{\partial\beta} - V_2\frac{\partial V_1}{\partial\beta}\right]_{\beta_1}^{\beta_2} d\alpha\,. \tag{3.7}$$

But by the boundary conditions, each [ ] term will vanish; consequently if $\mu_1 \neq \mu_2$ the integral on the left hand side must vanish.

To extend to the case when $\mu_1 = \mu_2$, $\lambda_1 \neq \lambda_2$, we write the left side of (3.5) as

$$\int_{\alpha_1}^{\alpha_2} f(\alpha) v_1(\alpha) v_2(\alpha) d\alpha \int_{\beta_1}^{\beta_2} v_1(\beta) v_2(\beta) d\beta - \\ - \int_{\beta_1}^{\beta_2} f(\beta) v_1(\beta) v_2(\beta) \int_{\alpha_1}^{\alpha_2} v_1(\alpha) v_2(\alpha) d\alpha \; . \tag{3.8}$$

Then the second and third integrals in this expression both vanish, so (3.5) holds when $\mu_1 = \mu_2$, $\lambda_1 \neq \lambda_2$, and the result is established.

Again, it should be noted that if $v_1 \equiv v_2$, the integral in (3.5) is not necessarily non-zero. However, if

(i) both paths of integration are parallel to either the real or the imaginary axis,

(ii) $v_1(\alpha)$ is either real or a pure imaginary on the path (α_1, α_2) and $v_1(\beta)$ is either real or pure imaginary on (β_1, β_2),

(iii) $f(\alpha)$, $f(\beta)$ are real and $f(\alpha) - f(\beta)$ is of one sign when α lies in (α_1, α_2) and β lies in (β_1, β_2),

then the integral

$$\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} \{v_1(\alpha)v_1(\beta)\}^2 \{f(\alpha) - f(\beta)\} \, d\alpha \, d\beta \neq 0 . \tag{3.9}$$

The conditions (i) - (iii) are satisfied in the problems (A) - (D) mentioned above.

Two further observations should be made at this point. First, if (α_1, α_2) and (β_1, β_2) denote the same path, then (3.5) is still true but is trivial, as may be seen by writing it in the form (3.8). In this case the integral in (3.9) vanishes in all circumstances.

Secondly, the results of this section still apply if any of the conditions $w(a) = 0$, etc. is replaced by $w'(a) = 0$, as is immediately obvious from (3.2) and (3.7).

4. Eigenfunction - expansions

From the orthogonality theorems given in §3 there follow immediately two formal eigenfunction expansion theorems. In the case when the eigenvalues are discrete, let us denote the eigenvalue-pairs by

$$\mu = \mu_i \qquad (i = 0, 1, 2, \dots) \tag{4.1a}$$

$$\lambda = \lambda_{ij} \qquad (j = 0, 1, 2, \dots)\ , \tag{4.1b}$$

and the corresponding eigenfunctions by

$$w = w_{ij}(z)\ . \tag{4.1c}$$

The index $\underline{i}$ can generally be expected to run through the values 0 to infinity; for the index $\underline{j}$, however, there are various possibilities. It may be (as in problem (A)) that j also varies from 0 to infinity; it may be (as in problems (B), (C)) that j runs only through a finite number of values (depending on $\underline{i}$), or in the extreme case (as in problem (D)) there is only a single $\underline{j}$ corresponding to each $\underline{i}$.

Then the formal expansion theorems are

I. If a function $F(z)$ is expressible as a single series

$$F(z) = \sum_j a_j\, w_{ij}(z) \tag{4.2}$$

(the summation being over all admissible values of j), then there follows immediately from (3.1) the formal relation

$$a_j \int \{w_{ij}(z)\}^2 dz = \int F(z) w_{ij}(z)\, dz \ , \tag{4.3}$$

where integration is over any of the three paths (a, b), (a, c), (b, c) .

This formula will, naturally, only determine the coefficients a_j if the integration processes are valid and the integral on the left is non-zero; conditions for the latter were discussed in §3 above.

II. If a function $F(\alpha, \beta)$ is expressible as a double series

$$F(\alpha, \beta) = \sum_i \sum_j A_{ij} w_{ij}(\alpha) w_{ij}(\beta) \ , \tag{4.4}$$

then from (3.5) we have similarly

$$A_{ij} \iint \{w_{ij}(\alpha) w_{ij}(\beta)\}^2 \{f(\alpha) - f(\beta)\}\, d\alpha\, d\beta =$$

$$\iint F(\alpha, \beta) w_{ij}(\alpha) w_{ij}(\beta) \{f(\alpha) - f(\beta)\}\, d\alpha\, d\beta \ , \tag{4.5}$$

where integration is over any two of the three paths (a, b), (a, c), (b, c) .

Again, the usefulness of this is subject not only to the existence of the expansion but also to the validity of the integration processes and the non-vanishing of the left hand integral.

In general, one can expect (in a given problem) the class of functions for which an expansion of type II holds to be considerably wider than the corresponding class for expansions of type I. For instance, in problem (C), an expansion of type I is possible only if $F(z)$ is effectively a polynomial of fixed degree (but otherwise arbitrary) while an expansion of type II is possible provided only that a certain simple transformation to new variables θ, ϕ say $F(\alpha, \beta) \neq G(\theta, \phi)$ is such that the resulting function $G(\theta, \phi)$ can be expressed as a double series of spherical harmonics. In some problems, however, the class of functions possessing a type - I expansion is still wide - e.g. in problem (A) it coincides with the class of functions $F(z)$ expressible as Fourier series.

One may conjecture that in the case of a "spectrum" continuous in λ or μ or both, analogous expressions may be possible using integral formulae in place of single or double series.

5. Real character of the eigenvalues

The orthogonality relations of §3 can be used to prove that under certain conditions the eigenvalues of λ, μ are all real.

Theorem

Denote by $u(z, \lambda, \mu)$ a solution of (1.1) such that $u(a, \lambda, \mu) = 0$, and by (α_1, α_2), (β_1, β_2) different value-pairs of (a, b), (a, c), (b, c).

Let (i) paths (α_1, α_2), (β_1, β_2) exist which are parallel to either the real or the imaginary axis and which do not pass through any singularity of the equation (1.1),

(ii) $u(z, \lambda, \mu)$ be either real or a pure imaginary on (α_1, α_2) and (β_1, β_2),

(iii) $f(\alpha) - f(\beta)$ be real and of one sign for α lying on (α_1, α_2) and β lying on (β_1, β_2).

Then the eigenvalues of the problem (1.1), (1.2) are all real.

Proof

The eigenvalues λ, μ are determined by the two equations

$$u(b, \lambda, \mu) = u(c, \lambda, \mu) = 0 \tag{5.1}$$

and by conditions (ii), these imply

$$u(b, \bar{\lambda}, \bar{\mu}) = u(c, \bar{\lambda}, \bar{\mu}) = 0\,. \tag{5.2}$$

Hence if λ_0, μ_0 be a complex eigenvalue-pair with an associated solution $u_0(z)$, $\overline{\lambda}_0$, $\overline{\mu}_0$ will be another eigenvalue pair with associated solution $\overline{u}_0(z)$. Then by the same reasoning as that which gave (3.7), with $\mu_1 = \mu_0$, $\mu_2 = \overline{\mu}_0$, $v_1 = u_0$, $v_2 = \overline{u}_0$, we obtain

$$(\mu_0 - \overline{\mu}_0) \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \{f(\alpha) - f(\beta)\} u_0(\alpha) u_0(\beta) \overline{u}_0(\alpha) \overline{u}_0(\beta) \, d\alpha \, d\beta = 0. \tag{5.3}$$

But the integrand is real and of one sign, so the integral is either real or a pure imaginary, and not zero; hence $\mu_0 - \overline{\mu}_0 = 0$, and μ_0 is real.

Further, λ_0 and $\overline{\lambda}_0$ are now eigenvalues belonging to the same' value of μ, so by applying a similar argument to the first integral relation (3.2) we prove that $\lambda_0 - \overline{\lambda}_0 = 0$, so λ_0 is also real.

<u>Corollary</u>

If a, b, c are real with $a < b < c$, $f(z)$, $g(z)$ are real for $a \leq z \leq c$, and $f(z)$ is monotone for $a \leq z \leq c$, and the equation has no singularities in $a \leq z \leq c$, then the eigenvalues are real.

For the conditions of the theorem are clearly all fulfilled in this case.

6. Integral relations and integral equations

In this section we give two integral relationships, from which can be derived integral equations satisfied by the solutions of (1.1), (1.2). The first of these is similar to the familiar integral equations for one-parameter eigenvalue problems, but the second is a non-linear equation of a rather unusual type.

Theorem I (First integral theorem)

Let (i) $w(z)$ be a solution of $w'' + \{\lambda + \mu f(z) + g(z)\}w = 0$, (6.1)

(ii) $G(z, z')$ be a solution of the partial differential equation

$$\frac{\partial^2 G}{\partial z^2} - \frac{\partial^2 G}{\partial z'^2} + [\mu\{f(z) - f(z')\} + g(z) - g(z')]G = 0, \quad (6.2)$$

such that $G(z, z')$ is analytic when z, z' lie in certain regions R, R' of the complex z, z'-planes respectively.

(iii) C be a path in the z'-plane, lying wholly within R' and such that

(α) $\left[G(z, z')\frac{dw(z')}{dz'} - w(z')\frac{\partial G(z, z')}{\partial z'} \right]$

has the same value at the two ends of C,

(β) $\int_C G(z, z')\, w(z')\,dz'$ exists for all z in R, and if the integral is singular, it converges uniformly with respect to z for all z in R.

Then

$$W(z) \equiv \int_C G(z, z')\, w(z')\, dz' \tag{6.3}$$

satisfies (6.1) for all $\underline{z}$ in $\underline{R}$.

Proof

By conditions (iii) differentiation of $W(z)$ under the integral sign is permissible, so

$$\frac{d^2W}{dz^2} + \{\lambda + \mu f(z) + g(z)\}\, W = \int_C \left[\frac{\partial^2 G}{\partial z^2} + \{\lambda + \mu f(z) + g(z)\} G\right] w(z')\, dz' =$$

$$= \int_C \left[\frac{\partial^2 G}{\partial z'^2} + \{\lambda + \mu f(z') + g(z')\}\, G\right] w(z')\, dz' , \qquad \text{(using (ii))}$$

$$= \left[\frac{\partial G}{\partial z'} w(z') - G(z, z')\frac{dw(z')}{dz'}\right]_C + \int_C [w''(z') + \{\lambda + \mu f(z') + g(z')\} w(z')]\, G dz'$$

(on integrating the product $w(z')\partial^2 G/\partial z'^2$ twice by parts),

$$= 0 ,$$

since the integrated term vanishes because of (iii) (α) and the integral because of (i). This proves the theorem.

Corollary (First integral equation)

Let $w(z)$, $w^*(z)$ be a pair of independent solutions of (6.1), such that $w(z)$ can be characterized by possessing a certain property P, and

$w^*(z)$ by possessing another property P^*, but a combination of w, w^* possesses neither property P nor P^*.

Then if (i) $G(z, z')$, as a function of z, has property P,

(ii) the function $W(z)$ defined by (6.3) is not identically zero,

then $w(z)$ satisfies the integral equation

$$w(z) = \theta \int_C G(z, z') w(z') dz' . \tag{6.4}$$

The proof of this is immediate when we remark that $W(z)$, being a solution of (6.1) must be of the form $W(z) = c\,w(z) + c^* w^*(z)$, where c, c^* are not both zero, and condition (i) implies $c^* = 0$.

Some observations must be made at this point regarding both the theorem and the corollary.

First, we have not imposed any boundary conditions on $w(z)$ or on $G(z, z')$. If we do add to $w(z)$, however, the restriction of satisfying the boundary conditions $w(a) = w(b) = 0$, and to G the conditions $G(z, a) = G(z, b) = 0$, and we take the path C as a suitable path joining $z = a$ to $z = b$, then this ensures that the integrated term does indeed vanish - i.e. condition (iii) (α) of the theorem is satisfied. A similar remark holds for a path joining $z = a$ to $z = c$ or $z = b$ to $z = c$.

Secondly, the theorem can be used, in certain circumstances, to derive an expression for the second solution of the differential equation

(6.1) in terms of the first solution. In the case of Mathieu's and Lamé's equations, for instance, this provides a very fruitful source of information about the second solution and its properties (2), (4, §2.6).

Thirdly, it should be noted that we have not yet suceeded in making a genuine reduction in the number of parameters involved in the problem. For the partial differential equation (6.2) which must be satisfied by G still contains the parameter μ, so that in general G itself will contain μ, while the parameter λ has been replaced in the integral equation (6.4) by a new unknown parameter θ.

Theorem II (Second integral theorem)

Let (i) $w(z)$ be a solution of (6.1),

(ii) $H(\alpha, \beta, \gamma)$ be a solution of the partial differential equation

$$\{f(\beta) - f(\gamma)\} \frac{\partial^2 H}{\partial \alpha^2} + \{f(\gamma) - f(\alpha)\} \frac{\partial^2 H}{\partial \beta^2} + \{f(\alpha) - f(\beta)\} \frac{\partial^2 H}{\partial \gamma^2} =$$

$$= [g(\alpha)\{f(\beta) - f(\gamma)\} + g(\beta)\{f(\gamma) - f(\alpha)\} + g(\gamma)\{f(\alpha) - f(\beta)\}] H, \quad (6.5)$$

such that $H(\alpha, \beta, \gamma)$ is analytic when α, β, γ lie in regions $R_\alpha, R_\beta, R_\gamma$ of the complex α-, β-, γ-planes respectively,

(iii) C_α, C_β be paths in R_α, R_β such that

$$\left[\frac{\partial H}{\partial \alpha} w(\alpha) - H(\alpha, \beta, \gamma) w'(\alpha) \right]_{C_\alpha} = 0, \quad (6.6a)$$

$$[\frac{\partial H}{\partial \beta} w(\beta) - H(\alpha, \beta, \gamma)\, w'(\beta)\,]_{C_\beta} = 0\,, \tag{6.6b}$$

(iv) $$W(\gamma) \equiv \int_{C_\alpha} \int_{C_\beta} \{f(\alpha) - f(\beta)\}\, H(\alpha, \beta, \gamma)\, w(\alpha)\, w(\beta)\, d\alpha\, d\beta \tag{6.7}$$

exists, and if singular converges uniformly with respect to γ when α, β, γ lie in $R_\alpha, R_\beta, R_\gamma$.

Then $W(z)$ is a solution of (6.1) .

<u>Proof</u>

Differentiation under the integral sign is valid by (iv) , so

$$W''(\gamma) = \int\int \{f(\alpha) - f(\beta)\} \frac{\partial^2 H}{\partial \gamma^2} w(\alpha)\, w(\beta)\, d\alpha\, d\beta\,,$$

hence

$$W''(\gamma) + \{\lambda + \mu f(\gamma) + g(\gamma)\}\, W =$$

$$= \int\int \{f(\alpha) - f(\beta)\} [\frac{\partial^2 H}{\partial \gamma^2} + \{\lambda + \mu f(\gamma) + g(\gamma)\} H]\, w(\alpha)\, w(\beta)\, d\alpha\, d\beta\,. \tag{6.8}$$

Now we use (6.5) to rearrange the integrand in (6.8) as

$$[\{f(\gamma) - f(\beta)\}\frac{\partial^2 H}{\partial \alpha^2} + \{f(\alpha) - f(\gamma)\}\frac{\partial^2 H}{\partial \beta^2} - F(\alpha, \beta, \gamma) +$$

$$+ \{\lambda + \mu f(\gamma) + g(\gamma)\}\{f(\alpha) - f(\beta)\}\, H(\alpha, \beta, \gamma)\,]\, w(\alpha)\, w(\beta)\,, \tag{6.9}$$

where $F(\alpha, \beta, \gamma)$ denotes the expression on the right hand side of (6.5) .

We now substitute (6.9) in (6.8), integrate the first term twice by parts with respect to α, and the second twice with respect to β; by (6.6a, b), the integrated terms vanish. Then we have

$$W''(\gamma) + \{\lambda + \mu f(\gamma) + g(\gamma)\} W =$$

$$= \int\int \{f(\gamma) - f(\beta)\} \{w''(\alpha) + g(\alpha) w(\alpha)\} w(\beta) H(\alpha, \beta, \gamma) \, d\alpha \, d\beta +$$

$$+ \int\int \{f(\alpha) - f(\gamma)\} \{w''(\beta) + g(\beta) w(\beta)\} w(\alpha) H(\alpha, \beta, \gamma) \, d\alpha \, d\beta +$$

$$+ \int\int \{\lambda + \mu f(\gamma)\} \{f(\alpha) - f(\beta)\} w(\alpha) w(\beta) H(\alpha, \beta, \gamma) \, d\alpha \, d\beta . \qquad (6.10)$$

Using, finally, the fact that $w(z)$ is a solution of (6.1) so that

$$w''(\alpha) + g(\alpha) w(\alpha) = -\{\lambda + \mu f(\alpha)\} w(\alpha) ,$$

etc., the right hand side of (6.10) reduces to

$$\int\int [\{f(\beta) - f(\gamma)\} \{\lambda + \mu f(\alpha)\} + \{f(\gamma) - f(\alpha)\} \{\lambda + \mu f(\beta)\} +$$

$$+ \{f(\alpha) - f(\beta)\} \{\lambda + \mu f(\gamma)\}] w(\alpha) w(\beta) H(\alpha, \beta, \gamma) \, d\alpha \, d\beta$$

$$= 0 \quad (\text{since the expression in } [\ ] \text{ vanishes identically}).$$

This proves the theorem. With the notation of Theorem I corollary, and using the same method of proof, we can add the corresponding corollary here

Corollary (Second integral equation)

Let $H(\alpha, \beta, \gamma)$, as a function of γ, have property P, and let the integral defining $W(\gamma)$ in (6.7) be not identically zero. Then $w(z)$ satisfies the integral equation

$$w(\gamma) = \phi \int_{C_\alpha} \int_{C_\beta} \{f(\alpha) - f(\beta)\} H(\alpha, \beta, \gamma) w(\alpha) w(\beta) \, d\alpha \, d\beta . \tag{6.11}$$

Again, it should be observed that no boundary conditions have been applied to $w(z)$. If we take C_α, C_β as suitable paths joining (α_1, α_2) and (β_1, β_2), where (α_1, α_2), (β_1, β_2) are different pairs of values from the three sets (a, b), (a, c), (b, c), then by imposing on $w(z)$ the boundary conditions (1.2) and on $H(\alpha, \beta, \gamma)$ the conditions $H(\alpha_1, \beta, \gamma) = H(\alpha_2, \beta, \gamma) = H(\alpha, \beta_1, \gamma) = H(\alpha, \beta_2, \gamma) = 0$, we ensure that the conditions (6.6a, b) for the vanishing of the "integrated terms" are satisfied.

It should be observed that we have here made a genuine reduction of our original problem to a one-parameter problem, for the partial differential equation (6.5) to be satisfied by the nucleus $H(\alpha, \beta, \gamma)$ is now completely independent of both λ and μ, so that the only remaining parameter is the ϕ of the integral equation (6.11). It is noteworthy that because of the non-linearity of (6.11), the value of ϕ depends on the normalisation of $w(z)$.

Integral equations of type (6.11) have been considered by Schmidt in (5).

This integral relation has been of considerable utility in dealing with problems (C) and (D); in these two problems the equation (6.5) is merely a transformation, respectively, of Laplace's equation and the wave equation in three dimensions. Among the applications made are (i) the obtaining of perturbation solutions for the low-frequency case of (D), (ii) finding the asymptotic behaviour of solutions $w(\gamma)$ for high frequencies or in certain regions of the γ-plane (both these applications were made by Malurkar in [3]) and (iii) expressions for the second solution of Lamé's equation in terms of Lamé polynomials [2]. A detailed study of the nuclei of an integral equation of type (6.11) for the problem (D) is in [1].

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